Calculating the Value of $\pi$

Written by Dave Didur

October 25, 2014 – What do you get if you divide the circumference of a jack-o’-lantern by its diameter? Pumpkin pi.

This is only funny if you know enough mathematics to understand that there is a fundamental relationship that is true for all circles: the ratio of the circumference (i.e. the perimeter, or distance around the outside) to the diameter (the distance across the centre) is a constant (i.e. it is the same value for all circles, regardless of size).

Mathematicians call this special value “pi” – and use the Greek letter $\pi$ (pronounced ‘pi’) – to represent that value.

$$\frac{\text{Circumference of a circle}}{\text{Diameter of a circle}} = \text{a constant}$$

$$\frac{C}{d} = \pi$$

This fact about circles has been known by mathematicians for thousands of years, but the specific value of this constant has not been known until recently. It was apparent, however, that it was not an exact number.

Measurements only produce rough approximations. You can try this yourself with your children. Look around the house and find different circle shapes such as different-sized jar lids, a Frisbee, a dinner plate, or a bowl. You’ll need a ruler to measure the diameters. If you don’t have a flexible measuring tape for measuring the circumferences, then a piece of string will do. Wrap the string around the outside of the circle, mark it with a pen, and then lay it out flat on a ruler. A ruler that is marked off into centimetres and millimetres is best because the measurement can be read directly as decimals.

I measured the round opening of my coffee filter basket:

- Diameter = 13.7 cm
- Circumference = 43.5 cm

Using long division (or a calculator), my measurements produce $C \div d \approx 3.237$.

Do this for many circular objects.

How close are the results? What is the value of $\pi$? How do mathematicians determine it? Read on!
The Geometric Period (2000 BC – 1650 AD)

Early civilizations considered circles and spheres to be “perfect shapes,” so they received a lot of attention and study. Evidence shows that 4,000 years ago Babylonians, Egyptians, Indians, Chinese and Greeks were aware that the ratio of the circumference to the diameter of circles was constant.

The Great Pyramid of Giza was completed by the Egyptians in approximately 2560 BC. Many believe that the dimensions of this structure incorporate some famous mathematical ratios (namely \(\pi\), the ratio of the circumference to the diameter of a circle, and \(\Phi\), the ratio of the length to the width of the Golden Rectangle).

For centuries no symbol was attached to the constant \(C : d\); awkward descriptive expressions were used instead – such as this medieval Latin phrase: quantitas in quam cum multificetur diameter, proveniet circumferencia (the quantity which, when the diameter is multiplied by it, yields the circumference). It wasn’t until 1706 that William Jones, a Welsh mathematician, introduced the symbol \(\pi\) (probably because the Greek word for ‘circumference’ was ‘\(\pi\varepsilon\rho\tau\varepsilon\rho\tau\alpha\)’) to represent the \(C : d\) ratio. I will make use of \(\pi\) in my historical discussions even though the symbol did not exist in those early years.

What kinds of estimates were used for \(\pi\)?

The Old Testament has two references (I Kings 7:23, written in approx. 550 BC, and Chronicles 4:2, written in approx. 450 BC) to physical measurements involving circles. I Kings 7:23 states, "Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits in height thereof; and a line thirty cubits did compass it round about." From these dimensions, the ratio of \(C : d = 30 : 10\), which yields a rough value of 3 for \(\pi\). Of course, the dimensions are given in whole numbers and should be taken as crude estimates of the actual shape.

Babylonians used the value \(\frac{3}{8} = 3.125\) for \(\pi\). In India, \(\sqrt{10}\) was used (approx. 3.162). The Rhind Papyrus (circa 1650 BC) is a famous historical artefact containing details of the mathematical knowledge of the early Egyptians. It includes this rule for computing the area of a circle: take away one-ninth of the diameter and then square the remainder:

\[
A = \left(1 - \frac{1}{9}\right)d^2 = \left(\frac{8}{9}d\right)^2 = \frac{64}{81}d^2 \quad \#1
\]

We now know that the area of a circle can be computed exactly using the formula \(A = \pi r^2\). Since the radius \(r\) is one-half of the diameter \(d\), we can rewrite this formula as:

\[
A = \pi r^2 = \pi \left(\frac{d}{2}\right)^2 = \pi \left(\frac{d^2}{4}\right) = \frac{\pi}{4}d^2 \quad \#2
\]
By setting the current formula (#2) equal to the Egyptian formula (#1), we can determine what value the early Egyptians used for pi:

\[ \frac{\pi d^2}{4} = \frac{64}{81} d^2 \]
\[ \frac{\pi}{4} = \frac{64}{81} \]
\[ \pi = 4 \times \frac{64}{81} \]
\[ \pi = \frac{256}{81} \]
\[ \pi = \left( \frac{16}{9} \right)^2 \]

This works out to be approximately 3.16.

In the 1st century, the Chinese used the ratio of the ‘celestial circle’ to the diameter of the earth, \( \frac{92}{29} \approx 3.172 \), as pi (later fined-tuned to \( \frac{142}{45} \approx 3.156 \)).

The Greeks were noted for important developments in mathematics, particularly in geometry (refer to my article Euclidean Geometry). Euclid, in approx. 300 BC, compiled the work of many of his predecessors and also included original material in a major treatise called Elements. Proposition 2 of Book XII states: circles are to one another as the squares on their diameters. The reference here is to the areas of circles. If we say that one circle has an area of \( A_1 \) and a diameter of \( d_1 \) and another has an area of \( A_2 \) and a diameter of \( d_2 \), then:

\[ \frac{A_1}{A_2} = \frac{(d_1)^2}{(d_2)^2} \]

As we saw earlier, the area of a circle can be expressed by the formula \( A = \frac{\pi d^2}{4} \).

A rearrangement of this formula produces \( \frac{A}{d^2} = \frac{\pi}{4} \), so we can see that Euclid was aware of the existence of a constant that was involved in the areas of circles - in addition to the constant involved in the circumference of circles.

The Greek mathematician Archimedes of Syracuse (287-212 BC) was the first person to develop a method (which was based on inscribed and circumscribed polygons) to compute the value of this constant. In principle, his technique allows for the calculation of \( \pi \) to any required degree of accuracy. Unfortunately, precise methods for the computation of square roots did not exist at that time, so Archimedes had to improvise with various techniques that allowed for close approximations. He was able to determine that \( \pi \) had a value that was between \( \frac{310}{71} \) and \( \frac{31}{7} \).

In terms of decimals, this means that he calculated that \( \pi \) was between 3.1408 and 3.1428 - approximately 3.14 (two decimal places of accuracy).
Archimedes’ method involved drawing regular-shaped polygons (equal-sided figures) that were inscribed inside a circle and circumscribed outside a circle. He began with 6-sided polygons (hexagons) as shown below.

As you can see, a circle is “trapped” between these inner and outer hexagons. The perimeter of the inner hexagon is smaller than the perimeter (or circumference) of the circle, whereas the perimeter of the outer hexagon is greater than the circumference of the circle. Archimedes used geometrical methods including the Pythagorean Theorem to compute the inner and outer perimeters, giving an upper and lower bound for the circumference of the circle. He began with a circle having a diameter $d = 1$, and since \( \frac{C}{d} = \pi \) his calculations for the circumference $C$ were actually estimations for the value of $\pi$.

Archimedes repeated these computations using 12-gons (a regular polygon with 12 sides), 24-gons, 48-gons and 96-gons. At each successive stage the number of sides was doubled. As the number of sides increased, the polygons tucked themselves closer and closer to the circle – making the perimeter calculations tighten around the value for pi.

These images, from the Wikipedia article on $\text{Pi}$, illustrate what happens as the number of sides increases for the inner and outer polygons. The 1st picture shows 5-gons (pentagons), the 2nd shows 6-gons (hexagons), and the 3rd shows 8-gons (octagons). The measurements of the perimeters of the inner and outer polygons get closer to each other. The perimeter (circumference) of the circle can be more closely estimated at each stage because its value is between the two results. By using a method that calculates the perimeters of the polygons, the circumference of the circle can be estimated.

\[
\frac{C}{d} = \pi, \text{ so if the circle has a diameter } = 1 \text{ then } C = \pi.
\]

Thus, the value of $\pi$ can be estimated.

I will detail an investigation that illustrates the technique of Archimedes, but using trigonometric (instead of geometric) methods. Trigonometric methods were not in existence at the time of Archimedes.
Before beginning the investigation to calculate the value of pi, I’ll review some trigonometric facts that are important to this approach.

**Trigonometry** is a branch of mathematics that originated from “triangle measurements”.

If the three angles in one triangle are equal to the three angles in another triangle, the triangles are said to be similar – even though their sizes are different. For example, in the illustration at the right \( \triangle QRP \cong \triangle ACB \) (i.e. the triangles are similar to each other because \( \angle Q = \angle A, \angle R = \angle C \) and \( \angle P = \angle B \)). In similar triangles, the ratios of corresponding sides are equal.

The triangle below has angles of 30°, 60° and 90°. Because it is a right-angled triangle, its sides satisfy the Pythagorean Theorem. You can verify that:

\[
(2)^2 = (\sqrt{3})^2 + (1)^2
\]

\[
4 = 3 + 1
\]

From the perspective of the 30° angle, the ratio of opposite side (1) to the hypoteneuse (2) is \( \frac{1}{2} \). Every other 30-60-90° triangle, regardless of its size, will have the same ratio (\( \frac{1}{2} \)) for the two corresponding sides, in relation to the 30° angle.

At the right, this ratio and other trig ratios are defined and calculated for a 30° angle.

Instead of always saying from the perspective of the 30° angle, the ratio of the opposite side to the hypoteneuse is \( \frac{1}{2} \) mathematicians use this expression to mean the same thing:

\[
\sin 30° = \frac{1}{2}
\]

This is called the **sine ratio** for 30°.

In like fashion, there are definitions for other side ratios in 90° triangles:

\[
\cos 30° = \frac{\sqrt{3}}{2}
\]

\[
\tan 30° = \frac{1}{\sqrt{3}}
\]

These are called the **cosine** and **tangent** ratios for a 30° angle. These are constants (just like pi). There is a sine, cosine and tangent ratio for every angle. Expressed as decimals instead of fractions:

\[
\sin 30° = 0.5
\]

\[
\cos 30° = 0.8660254…
\]

\[
\tan 30° = 0.5773502…
\]

Calculators and spreadsheets have the decimal values of the trig ratios built into them.

It is important to know that angles can be measured in degrees or in another unit called **radians** (based on the value of pi). The relationship between these units is \( 180° = \pi \) radians. Spreadsheets tend to use radian measurements; calculators can be set into either mode. For more information on angle measurements and radians, check the Resources section.
Calculating the Value of Pi

A circle, with centre C and diameter = 1, is drawn (in broken blue). AC and DC are radii of the circle, so their lengths are 0.5 (half of the diameter). The brown regular hexagon ADFKNP is inscribed inside the circle (this means that its six vertices are on the circle). The green regular hexagon MEGHJL is circumscribed around the circle (each of its sides is tangent to the circle). Because these are both regular hexagons, each hexagon has six equal sides.

Perimeter of inner hexagon = 6 X AD
Perimeter of outer hexagon = 6 X EG

A geometric theorem states: a radius drawn to the mid-point of a chord is perpendicular to the chord. B is the mid-point of chord AD, so AB = ½ of AD, and EC is perpendicular to AD (this means that angle ABC = 90°).

Another geometric theorem states: a radius drawn to the point of contact of a tangent is perpendicular to that tangent. Thus, angle CDE = 90°. D is the mid-point of EG, so ED = ½ of EG.

Perimeter of inner hexagon = 6 X AD = 6 X (2 X AB) = 12 X AB
Perimeter of outer hexagon = 6 X EG = 6 X (2 X ED) = 12 X ED

The perimeter of the circle (i.e. the circumference) is a value between these two perimeters.

Trigonometry allows us to calculate the inner and outer perimeters (Archimedes did this with geometric methods).

**Inner Hexagon**

In triangle ABC, \(\angle ABC=90^\circ\) and \(\angle ACB=30^\circ\)

\[
\sin 30^\circ = \frac{AB}{AC} = \frac{AB}{0.5}
\]

But \(\sin 30^\circ = 0.5\) (see previous discussion).

Thus, \(\frac{AB}{0.5} = 0.5\)

\(AB = 0.25\)

Perimeter = 12 X AB
= 12 X 0.25
= 3.0

**Outer Hexagon**

In triangle EDC, \(\angle EDC=90^\circ\) and \(\angle ECD=30^\circ\)

\[
\tan 30^\circ = \frac{ED}{EC} = \frac{ED}{0.5}
\]

But \(\tan 30^\circ = 0.5773502…\) (see previous discussion).

Thus, \(\frac{ED}{0.5} \approx 0.5773502\)

\(ED \approx 0.5 \times 0.5773502\)

\(ED \approx 0.2886751\)

Perimeter = 12 X ED
\[
\approx 12 \times 0.2886751
= 3.4641012
\]

Perimeter of inner hexagon < Circumference of circle < Perimeter of outer hexagon

\(3 < \text{Circumference of circle} < 3.4641\)

Since \(\pi = \frac{C}{d} = \frac{C}{1} = C\) for this circle of diameter 1, we get:

\(3 < \pi < 3.4641\)

This roughly indicates that the value of pi lies between 3 and 3.4641.
Archimedes continued his calculations using polygons with more and more sides – specifically 12-gons, 24-gons, 48-gons and 96-gons. It was painstaking work using the mathematical tools of his day – and he had to use estimates for calculating large square roots – so the accuracy of his results were limited. Finally, for the 96-gon, he determined that $\frac{223}{71} < \pi < \frac{22}{7}$, a result that was accurate only to two decimal places for $\pi$ (i.e. $\pi \approx 3.14$), but significantly more accurate than earlier approximations.

### Generalizing & Extending this Approach

Using the logic outlined in the box at the right, the perimeters of the inner and outer n-gons can be generalized:

Perimeter of inner n-gon = $n \times \sin \frac{180^\circ}{n}$

Similarly, the perimeter of outer n-gon = $n \times \tan \frac{180^\circ}{n}$

For spreadsheets, we must use radian measurement:

Perimeter of inner n-gon = $n \times \sin \frac{\pi}{n}$

Perimeter of outer n-gon = $n \times \tan \frac{\pi}{n}$

Using a spreadsheet, such as Excel, we can set up three columns and use these formulas. Each row represents a different n-gon – each one having twice as many sides as the previous polygon.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Sides (n)</td>
<td>Perimeter of inner n-gon</td>
<td>Perimeter of outer n-gon</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>=A2*SIN(PI()/A2)</td>
<td>=A2*TAN(PI()/A2)</td>
</tr>
<tr>
<td>3</td>
<td>=2*A2</td>
<td>=A3*SIN(PI()/A3)</td>
<td>=A3*TAN(PI()/A3)</td>
</tr>
<tr>
<td>4</td>
<td>=2*A3</td>
<td>=A4*SIN(PI()/A4)</td>
<td>=A4*TAN(PI()/A4)</td>
</tr>
<tr>
<td>5</td>
<td>=2*A4</td>
<td>=A5*SIN(PI()/A5)</td>
<td>=A5*TAN(PI()/A5)</td>
</tr>
<tr>
<td>6</td>
<td>=2*A5</td>
<td>=A6*SIN(PI()/A6)</td>
<td>=A6*TAN(PI()/A6)</td>
</tr>
</tbody>
</table>

It is not mathematically correct to be using the value of pi in order to estimate the value of pi – which is done in these formulas in order to convert angle measurements from degrees into radians. I am doing this simply to illustrate the approach used by Archimedes.

The results of this spreadsheet are shown on the next page.
Archimedes used only geometric methods. The video “Finding Pi by Archimedes’ Method” (in the Resources section) illustrates an approach that uses only geometric methods (the Pythagorean Theorem). It also shows why Archimedes used polygons of sides 6, 12, 24, 48, and 96 – doubling the number of sides in each step. A spreadsheet for this geometric approach is created at the end of the video to perform all of the calculations that are illustrated.

The spreadsheet for my trigonometric example produces the values shown below.

<table>
<thead>
<tr>
<th>Sides (n)</th>
<th>Perimeter of inner n-gon</th>
<th>Perimeter of outer n-gon</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>3</td>
<td>3.464101615</td>
</tr>
<tr>
<td>12</td>
<td>3.105828541</td>
<td>3.21590309</td>
</tr>
<tr>
<td>24</td>
<td>3.132628613</td>
<td>3.159659942</td>
</tr>
<tr>
<td>48</td>
<td>3.139350203</td>
<td>3.146086215</td>
</tr>
<tr>
<td>96</td>
<td>3.141031951</td>
<td>3.1427146</td>
</tr>
<tr>
<td>192</td>
<td>3.141452472</td>
<td>3.14187305</td>
</tr>
<tr>
<td>384</td>
<td>3.141557608</td>
<td>3.141662747</td>
</tr>
<tr>
<td>768</td>
<td>3.141583892</td>
<td>3.141610177</td>
</tr>
<tr>
<td>1536</td>
<td>3.141590463</td>
<td>3.141597034</td>
</tr>
<tr>
<td>3072</td>
<td>3.141592106</td>
<td>3.141593749</td>
</tr>
<tr>
<td>6144</td>
<td>3.141592517</td>
<td>3.141592927</td>
</tr>
<tr>
<td>12288</td>
<td>3.141592619</td>
<td>3.141592722</td>
</tr>
<tr>
<td>24576</td>
<td>3.141592645</td>
<td>3.141592671</td>
</tr>
<tr>
<td>49152</td>
<td>3.141592651</td>
<td>3.141592658</td>
</tr>
<tr>
<td>98304</td>
<td>3.141592653</td>
<td>3.141592655</td>
</tr>
</tbody>
</table>

For the 96-gon case (the last step in Archimedes’ calculations), this approach determines that 3.14103 < \( \pi \) < 3.14271. Notice that we would have to double the number of sides fourteen times from \( n=6 \) to get a 98304-gon before the value of \( \pi \) is calculated to approximately 8 decimal places of accuracy as \( \pi \approx 3.14159265 \).

Liu Hui was the first Chinese mathematician to provide a rigorous algorithm for calculating \( \pi \) to any accuracy. His approach (in the 3rd century AD) was similar to that of Archimedes, but he used calculations based on the areas of inner and outer n-gons. Archimedes had shown that the area of a circle could be computed using the formula \( A = \pi r^2 \), so a circle with a diameter = 2 (i.e. with a radius = 1) would have an area equal to \( \pi \). Liu Hui used this fact and calculated the areas of inner and outer n-gons at each stage.

Area of inner n-gon < Area of a circle of radius 1 < Area of outer n-gon
Area of inner n-gon < \( \pi \) < Area of outer n-gon

His calculations with a 96-gon provided an accuracy of five digits for \( \pi \approx 3.1416 \). Later Zu Chongzhi (430-501 AD) from China, also using polygons, calculated:

\[ 3.1415926 < \pi < 3.1415927. \]

This type of geometric approach was used until the 17th century.
The Classic Period (1650 - 1950 AD) – Analytical Methods

Great mathematicians such as Isaac Newton (1643 – 1727) and James Gregory (1638 – 1675) developed series expansions that could be used to approximate inverse trigonometric functions.

**Trigonometric functions** such as sine, cosine and tangent act on angles and produce decimal values. For example, \( \sin 30^\circ = \frac{1}{2} = 0.5 \) and \( \tan 45^\circ = \frac{1}{1} = 1.0 \)

**Inverse trig functions** act on a decimal or fraction value and produce an angle. For example, \( \arcsin \left( \frac{1}{2} \right) = \frac{\pi}{6} \) and \( \arctan(1.0) = \frac{\pi}{4} \) (the angle is in radians; \( \pi \) radians = 180°).

Newton is credited with developing an infinite series that approximates the arcsine function:

\[
\arcsin(x) = x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2 \cdot 2} \frac{x^5}{5} + \frac{1}{2 \cdot 4 \cdot 6} \frac{x^7}{7} + \ldots
\]

Since \( \arcsin \left( \frac{1}{2} \right) = \frac{\pi}{6} \), let \( x = \frac{1}{2} \) on the right side of this expansion and replace \( \arcsin \left( \frac{1}{2} \right) \) on the left side by \( \frac{\pi}{6} \):

\[
\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 2} \left( \frac{1}{2} \right)^3 + \frac{1}{2 \cdot 4 \cdot 6} \left( \frac{1}{2} \right)^5 + \frac{1}{2 \cdot 4 \cdot 6 \cdot 7} \left( \frac{1}{2} \right)^7 + \ldots
\]

Multiply both sides by 6 and the result is a formula that can be used to approximate pi. It is an infinite series, so greater accuracy is obtained as more terms are used.

Using just one term: \( \frac{\pi}{6} \approx \frac{1}{2} \), which gives \( \pi \approx 3.0 \).

Using 2 terms: \( \frac{\pi}{6} \approx \frac{1}{2} + \frac{1}{2 \cdot 2} \cdot \frac{1}{2} \approx \frac{1}{2} + \frac{1}{48} = \frac{25}{48} \),

which yields \( \pi \approx 6 \times \frac{25}{48} = \frac{25}{8} = 3.125 \).

Using 3 terms, we get

\[
\frac{\pi}{6} \approx \frac{1}{2} + \frac{1}{2 \cdot 2} \left( \frac{3}{1280} \right) = 0.523177
\]

\( \pi \approx 6 \times 0.523177 = 3.139063 \).

Using 4 terms, we get \( \pi \approx 3.141155 \).

Gregory developed the following arctangent series expansion:

\[
\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots
\]

Since \( \arctan(1) = \frac{\pi}{4} \), let \( x = 1 \) on the right side of this expansion and replace \( \arctan(1) \) on the left side by \( \frac{\pi}{4} \):

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots
\]

This is a much simpler series than the arcsine version on the left, but the terms alternate in sign (positive, negative, positive, negative…) so our term by term calculations produce results that get larger, then smaller, then larger, then smaller. Mathematicians describe this by saying that the series takes longer to converge to a good approximation for pi. Using one term: \( \pi \approx 4 \)

Using 2 terms: \( \pi \approx 2.666666 \). Using 3 terms: \( \pi \approx 3.466666 \). Using 4 terms: \( \pi \approx 2.895238 \). Using 5 terms: \( \pi \approx 3.339682 \). Using 6 terms: \( \pi \approx 2.976046 \). Using 7 terms: \( \pi \approx 3.283738 \). Using 8 terms: \( \pi \approx 3.017071 \). Using 9 terms: \( \pi \approx 3.252365 \). Using 10 terms: \( \pi \approx 3.041839 \). Using 11 terms: \( \pi \approx 3.232315 \). Using 12 terms: \( \pi \approx 3.058402 \). Using 13 terms: \( \pi \approx 3.218402 \). Using 14 terms: \( \pi \approx 3.070254 \). Using 15 terms: \( \pi \approx 3.208185 \).
The arcsine series expansion converges very quickly to pi – reaching 16 decimal places of accuracy after just 22 terms. The arctangent series expansion converges very slowly – taking over 300 terms to achieve 2 decimal places of accuracy!

Other mathematicians, such as Leonhard Euler (1707-1783), produced several additional infinite series. Two are given here:  
\[
\frac{\pi}{2} = 1 + \frac{1}{3} + \frac{1}{3} + \frac{2}{3} \ldots \\
\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots
\]

The Indian mathematical prodigy Srinivasa Ramanujan (1887-1920) produced many accurate, albeit complicated, infinite series approximations for pi. The following is one of them:  
\[
\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1}{3}\right)^3 - 13\left(\frac{1}{4}\right)^3 + \ldots
\]

In an earlier article, I described numbers called **Continued Fractions**. William Brouncker (1620-1684) produced this continued fraction involving pi:  
\[
\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \ldots}}}}
\]

It is not known how he came up with this expression. Continued fraction expressions for pi are not very useful for computing the value of pi, but they are useful for producing rational approximations for pi. The following is a classic continued fraction for pi:  
\[
\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{292 + \frac{1}{1 + \ldots}}}}
\]

In a more compact notation this can be written as  
\[
\pi = [3; 7, 15, 1, 292, 1, 1, 2, 13, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, \ldots]
\]

As a first approximation, \( \pi \approx 3 \) 
As a 2nd approximation, \( \pi \approx 3 + \frac{1}{7} = \frac{22}{7} \) (one of the most popular rational approximations for \( \pi \)) 
As a 3rd approximation, \( \pi \approx 3 + \frac{1}{7 + \frac{1}{15}} = 3 + \frac{1}{106} = 3 + \frac{15}{106} = 3 + \frac{333}{106} \) (more accurate than \( \frac{22}{7} \)) 
As a 4th approximation, \( \pi \approx 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{16}}} = 3 + \frac{1}{355} = 3 + \frac{16}{355} = \frac{333}{106} \) (more accurate than \( \frac{333}{106} \)) 
As a 5th approximation \( \pi \approx \frac{103993}{33102} \) and as a 6th approximation \( \pi \approx \frac{104348}{33215} \) (very accurate!)
The Modern Period (1950 AD onwards) – Computer Methods

The first computer to calculate pi was ENIAC (Electronic Numerical Integrator And Computer) in 1949, programmed by John von Neumann and his associates. The 2,037 digits of pi which were produced took 70 hours to compute.

For over two decades, Yasumada Kanada at the University of Tokyo was a key figure in the computation of pi. Using supercomputers and high speed iterative algorithms, he computed 201,326,000 decimal places in 1988, 6 billion digits in 1995, 17 billion digits in 1996, and 209 billion digits in 1999. In 2002, Kanada and his team calculated 1.24 trillion digits of pi in 400 hours using a powerful supercomputer called the Hitachi SR 8000 - capable of performing two trillion calculations per second!

In 2010, using a specially built $18,000 home computer, Shigeru Kondo and Alexander Yee computed five trillion digits of pi in a period of 90 days. As of 2013, pi has been computed to over 12 trillion digits!

Do we really need over 12,000,000,000,000 digits for pi? No.

- Scientific applications generally require no more than 40 digits of \( \pi \).
- If a value of \( \pi \) rounded off to 9 decimal places was used to calculate the circumference of the earth, the result would be accurate to within 0.000000016%.
- Only 39 decimal places of \( \pi \) would be necessary to calculate the circumference of a circle that would encircle the known universe – with an error that would be less than the radius of a proton!

The primary motivation for the extensive computations of the value of \( \pi \) to so many digits is the human desire to break records. For practical purposes, the trillions of iterative calculations involved in determining the value of \( \pi \) are used to test super-computers and high-precision multiplication algorithms.

Pi to one hundred decimal places:

3.141592653589793238462643383279502884197169399375105
8209749445923078164062862089986280348253421170679…
Having Fun With Pi

Do you like palindromes? Here’s one relevant to this topic:
“I PREFER PI”

If you enjoy π then you may wish to partake in the activities surrounding the annual PI DAY which is celebrated on March 14th (March = 3rd month, so this is 3 14). Check out the internet for more information as the date draws near. Some links are given below.

By the way, Albert Einstein was born on “Pi Day” in 1879.

Investigate the “Fun Facts” websites listed in the Resources section to discover other interesting facts related to this special number π.

The Guinness-recognized record for remembered digits of pi is 67,890 digits, held by Lu Chao, a 24-year-old graduate student from China. It took him 24 hours and 4 minutes to recite to the 67,890th decimal place of pi without an error.

No one has proven that the digits of pi aren’t normally distributed, so people generally assume that they are. The first million decimal places of pi consist of 99,959 zeros, 99,758 1’s, 100,026 2’s, 100,229 3’s, 100,230 4’s, 100,359 5’s, 99,548 6’s, 99,800 7’s, 99,985 8’s, and 100,106 9’s. Those statistics seem quite equal for all digits.

Spock thwarts an evil computer in the Star Trek episode “Wolf in the Fold” by telling it to “compute to the last digit the value of pi.”

This article is the eighth of a series of mathematics articles published by CHASA.

Marvellous Mathematics – Introduction
Euclidian Geometry – Article #1
Non-Euclidean Geometry – Article #2
Rational Numbers – Fractions, Decimals and Calculators – Article #3
Continued Fractions – Article #4
Introduction to Fractals: The Geometry of Nature – Article #5
Solving Algebraic Equations (One Variable) – Article #6
Solving Systems of Equations in Two Variables – Article #7

CHASA has received many communications from concerned parents about the difficulties their children are having with the math curriculum in their schools as well as their own frustration in trying to understand the concepts - so that they can help their children. The intent of these articles is to not only help explain specific areas of history, concepts and topics in mathematics, but to also show the beauty and majesty of the subject.
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Resources

Calculating Pi with Real Pies – Video (This is fun to watch!)

What is Pi and How Did it Originate? – Scientific American

Phi, Pi and the Great Pyramid of Egypt at Giza – Interesting facts about the dimensions of the pyramid in relation to the numbers Pi and Phi

The History of Pi – Video

William Jones and his Circle: the Man who Invented Pi – from History Today

How Pi is Calculated – Video (historical perspective)

Archimedes was a great thinker. At one time, while getting into his bath, he contemplated the overflow of water caused by lowering himself into the tub. “Eureka!” he exclaimed when he realized that his body was displacing an equal volume of water. This concept is known as Archimedes’ Principle. He invented a device (which we call Archimedes’ Screw) that could lift water from low-lying regions to higher areas – useful for irrigation and as a bilge pump in ships. His mathematical works focussed on studies related to circles, spheres and other geometric shapes. Besides calculating the value of pi, he determined that \( \pi r^2 \) was the area of a circle, \( \frac{4}{3} \pi r^3 \) was the volume of a sphere and \( 2\pi r^2 \) was the volume of a cylinder. He was killed by a Roman soldier during the siege of Syracuse. His tomb bore a sculpture with a sphere and a cylinder.

Finding Pi by Archimedes’ Method – Video (a geometric approach to illustrate Archimedes’ method using polygons approaching the circumference from the interior only)

Liu Hui’s Algorithm for Calculating Pi – Based on areas of N-gons

Where Does \( A = \pi r^2 \) Come From? – Deriving the area formula for a circle from the circumference formula \( C = \pi d \) (or \( C = 2 \pi r \))
Angle Measurement – degrees and radians

Infinite Expansions for Pi – Inverse trig functions and continued fractions examples

Pi and its Computation Through the Ages – Two charts are given: one that details computations of pi before the age of computers, and a 2nd that details various computer-related calculations

Collection of Approximations for Pi – Using rational, continued fractions, square roots, etc.

The First Computer to Calculate Pi – ENIAC

Calculating Pi with Computer or Pencil – Using arctangent formulas in C++

Calculating Pi to 5 Trillion Digits – Record set by a personal computer in 2010

Pi Facts – Celebrating Pi Day 2014

70 Fun Facts about Pi – Everything you wanted to know about pi but were afraid to ask 😊

Eight Pi Day Activities for Elementary School Classrooms – at 4mulaFun

Free Pi Day Activities – Printable sheets, books to read, activities, jokes, games, fun

Pi Day Resources – for teachers and parents (at MathMovesU)

Pi Day Worksheets – primary and intermediate activities (at SuperTeacherWorksheets)

Buffon’s Needle Problem – MathIsFun activity based on a classic experiment involving pi

Buffon’s Needle Simulation – drop needles and get immediate results (College of Education, Illinois)