Trial and Error as a Legitimate Form of Problem-Solving (PART 1)

Written by Dave Didur

January 15, 2015 -- When I was a young lad I was quite interested in puzzles. I acquired a large number of puzzle books and spent many happy (sometimes frustrating) hours working on the challenges while sitting in the back seat of the car on family vacations or during quiet times at home – much like many people do with crosswords or Sudokus today. To me they were fun. Here’s a problem that I remember well. Try to answer it yourself before reading the solution that follows.

PROBLEM #1: The Knockout Tournament

Eight teams are entered into a straight elimination tournament. When a team loses, it is out. The team that wins all of its matches is the champion. How many matches must be played in order to determine the winner?

The original problem used 78 teams – but I’ve simplified it. Cleverly I drew up the typical kind of schedule used for basketball tournaments during NCAA March Madness – and then counted the number of matches on the chart. As you see, the answer is seven.

The answer for my 78 team question was 77. If there had been 163 teams, the answer would have been 162. The solution is so simple: if there are x teams in an elimination tournament, then (x-1) of the teams must lose in order to declare a champion! That requires (x-1) games. Imagine what my chart looked like for 78 teams – with byes included! Logic provided the simplest solution. My brute force method had given me the correct answer, but the simple elegance of the logical solution showed me that I had a lot to learn about problem-solving.
Working on puzzles, I learned that there are many different ways to solve problems. When we think about mathematical problems we usually have visions of equations dancing in our heads – things like $5x - 2 = x + 10$ or $6x^2 - x - 12 = 0$ or $(x + 1)\frac{dy}{dx} = x - y$. The first equation (a linear equation) has one solution ($x = 3$), the second (a quadratic equation) has two ($x = \frac{3}{2}$ and $x = -\frac{4}{3}$), and the third (a differential equation) has an infinite number of solutions of the form $y = \frac{x^2 + C}{2x + 2}$ where $C$ can be any constant. In school we are taught many techniques for solving such algebraic equations, beginning in elementary school and continuing through high school and into university. These skills are needed by mathematicians, scientists and engineers – but they are not required for everybody. Time should be devoted to thinking skills, logic and alternative types of problems. In fact, as Sudoku lovers know, sometimes a point is reached where trial and error is needed instead of straight logic. In this article (and in Part 2 next month), I intend to demonstrate problems of a logical nature, and focus on the importance of trial and error as a legitimate method of solution.

### PROBLEM #2: Magic Squares

A magic square is an array of different natural numbers (i.e. 1, 2, 3, 4,...) in which the sum of all numbers in each row, column and long diagonal is the same. Create a 3 X 3 magic square using the first nine natural numbers.

**LOGIC:** The digits are 1, 2, 3, 4, 5, 6, 7, 8, 9. It seems logical to put the middle digit (the 5) in the centre of the square.

Place the remaining 8 digits in the grid so that each row, column, and the two diagonals all have the same sum.

![Magic Square](image)

**LOGIC:** Perhaps the first digit (1) should be paired with the last digit (9) in the same line – giving a sum of 15. In the same way, the 2nd digit (2) should be paired with the 2nd last digit (8), the 3 with the 7, and the 4 with the 6.

**TRIAL & ERROR:** Try these pairings in different locations until you get it right!

By the way, these types of puzzles aren't just limited to squares. To the left is a picture of three intersecting circles. The numbers from 1 to 6 have been placed on the points of intersection in such a way that the sum of the four numbers on any circle is the same (in this case, 14).

**CHALLENGE:** Place the numbers from 1 to 12 on the points of intersection of the four circles so that sum of the six numbers on each circle is the same.
The process of manipulating numbers and signs so as to produce a required answer is called “cooking the books.” Tempting the solver into following this ‘evil’ practice, I invite you to join in the following activity. Fiddle the arithmetic to produce the predetermined answer.

You have five 2s to work with: 2 2 2 2 2. You also have four arithmetic signs: + - X ÷. In every challenge you must place EACH of the four signs in the four spaces between the five 2s so that the arithmetic expression works out to equal the given answer. You may put brackets around groups of calculations, as desired, in order to affect the order of operations. For example, $2 + 2 \times 2 = 6$ because multiplication is performed before addition. But $(2 + 2) \times 2 = 8$ because calculations inside of brackets are always performed first. Your four challenges are given below.

Remember the Order of Operations: “BEDMAS”

1. Brackets – expressions enclosed within brackets are evaluated first (using the order of operations)
2. Exponents are evaluated next (we don’t have exponents in this puzzle, so we don’t need to concern ourselves about them)
3. Division AND Multiplication are performed next – in the order in which they are encountered (from left to right)
4. Addition AND Subtraction are performed last of all – in the order in which they are encountered from left to right

I’ll do one challenge to illustrate typical solutions. In order to come up with answers you must definitely use a trial & error approach!

EXAMPLE: Try to produce the answer 4 (i.e. $2 \square 2 \square 2 \square 2 \square 2 = 4$).

Here’s one solution: $(2 \div 2 + 2) \times 2 - 2 = 4$

It works out as $(1 + 2) \times 2 - 2 = 3 \times 2 - 2 = 6 - 2 = 4$

Here’s another: $(2 - 2) \div 2 + 2 \times 2 = 4$

It works out as $0 \div 2 + 2 \times 2 = 0 + 2 \times 2 = 0 + 4 = 4$

These are your challenges: 

(A) $2 \square 2 \square 2 \square 2 \square 2 = 0$ There are 2 possibilities.
(B) $2 \square 2 \square 2 \square 2 \square 2 = 1$ There are 2 possibilities.
(C) $2 \square 2 \square 2 \square 2 \square 2 = 3$ There are 2 possibilities.
(D) $2 \square 2 \square 2 \square 2 \square 2 = 2$ There are 16 possibilities.

How many can you find?

Write down a four-letter word that is the same read forwards, backwards or upside down.

This is not a mathematical problem – but it does illustrate the use of logic and trial & error.
LOGIC: It is a 4-letter word. It reads the same forwards and backwards, so the first letter must be the same as the last letter, and the 2nd must be the same as the 3rd. The word must be of the form ABBA (like the musical group ABBA).

LOGIC: The word reads the same upside down. Thus, all of the letters must be identical right-side up or upside-down. Going through the alphabet we find there are seven possible letters: H I N O S X Z

LOGIC: To be a word, it is likely that a vowel and a consonant are involved.

TRIAL & ERROR: Select one vowel and one consonant at a time – and try them out. Here are all of the possibilities:

- HIIH  IHII  HOOH  OHHO
- NIIN  INNI  NOON  ONNO
- SIIS  ISSI  SOOS  OSSO
- XIIX  IXXI  XOOX  OXXO
- ZIIZ  IZZI  ZOOZ  OZZO

You may have arrived at the answer without listing everything: NOON

PROBLEM #5: Alphametics and the Like Reference 3

The following type of mathematical problem (called an "alphametic") uses letters in place of digits. Each different letter represents a unique digit. If an E stands represents the number 5, then every E is a 5 – and no other letter is a 5. The best puzzles have WORDS in them, set up as additions, subtractions, multiplications or divisions. Here are a few to wile away the hours!

1) Addition:         2) Division:         3) Multiplication:  4) Addition:         5) Addition:
L O S E         D O         F U N         THE         X M A S
S E A L         D O         F L Y         I N         S E V E N         M A I L
S A L E S         F I         * * *         S E V E N         E A R L Y
D R Y         * * *         T E A S E R         P L E A S E
D R Y         F A C T

In 1914, this ‘classic’ addition problem was posed: SEND MORE MONEY. I’ll work it through with you.

S E N D
M O R E
M O N E Y

LOGIC: The M must be a “1” because it is a carry from the addition of S and M (plus the possibility of a carry over from the addition of the 2nd column).

LOGIC: How can S + 1 be 10 or more in order to produce the carry of 1? If S = 9, then S + 1 + carry = 9 + 1 + carry, which can only equal 10 or 11.

LOGIC: Since “O” = 0, the 2nd column cannot have a value for E large enough to make a carry into the 1st column unless E = 9 and a carry came from the 3rd column, but that would make a sum = 10 – and N cannot be 0 (it is already assigned). Thus, there is NO CARRY from E + 0, meaning that S must be 9.
LOGIC: Examine the 2nd column: \( E + 0 + \text{carry} = N \).

If there was no carry from the 3rd column, then \( E = N \), which is impossible because each letter represents a different digit. Thus, there is a carry over and that makes \( N \) one greater than \( E \) (i.e. \( N = E + 1 \)).

LOGIC: Examining the 3rd column, the addition is \( N + R + \text{carry} = 10 + E \) (the 10 part of the sum represents the “1” which is carried into the 2nd column).

*Time for some algebra! Substitute the blue equation into the green one…*

\[
\begin{align*}
N + R + \text{carry} &= 10 + E \\
(E + 1) + R + \text{carry} &= 10 + E \\
R + \text{carry} &= 10 + E - E - 1 \\
R + \text{carry} &= 9
\end{align*}
\]

If there is no carry, this makes \( R = 9 \) – but \( S = 9 \) so this is not possible. Thus, there must be a carry from the 4th column, and \( R = 8 \).

LOGIC: Now examine the 4th column addition: \( D + E = 10 + Y \) (we determined that there must be a carry over from this column). \( Y \) cannot be 0, 1, 8 or 9.

*The sum of \( D \) and \( E \) must be 12, 13, 14, 15, 16 or 17 – and the only digits left to use are 2, 3, 4, 5, 6 and 7. Let’s try different sums for \( D + E \).*

The largest *trial & error* part of the solution has arrived!

\[
\begin{align*}
2 + 3 & \text{ isn’t big enough} \quad \text{(*the sum must be greater than 11*)} \\
2 + 4 & \text{ isn’t big enough.} \quad 3 + 4 \text{ isn’t big enough.} \\
2 + 5 & \text{ isn’t big enough.} \quad 3 + 5 \text{ isn’t big enough.} \\
2 + 6 & \text{ isn’t big enough.} \quad 3 + 6 \text{ isn’t big enough.} \\
2 + 7 & \text{ isn’t big enough.} \quad 3 + 7 \text{ isn’t big enough.} \\
4 + 5 & \text{ isn’t big enough.} \quad 5 + 6 \text{ isn’t big enough.} \\
4 + 6 & \text{ isn’t big enough.} \quad 5 + 7 \text{ is big enough.} \\
4 + 7 & \text{ isn’t big enough.} \quad 6 + 7 \text{ is big enough.}
\end{align*}
\]

Examine the only possible cases:

\[
\begin{align*}
5 + 7 &= 12 \quad \text{which makes} \quad Y = 2. \quad \text{Possible.} \\
6 + 7 &= 13 \quad \text{which makes} \quad Y = 3. \quad \text{Possible.}
\end{align*}
\]

The only possible cases for \( D \) and \( E \) are 5 and 7 or 6 and 7. Thus, one of the letters must be a 7.

LOGIC: \( E \) cannot be 7 because \( E + 1 = N \) (2nd column result) – making \( N \) equal to 8, and 8 is already the value of \( R \). So \( D \) must be 7.

LOGIC: The logic wraps up by realizing that our 2nd column relationship was \( E + 1 = N \), so \( \bar{E} = 5 \) and \( N = 6 \). If \( E = 5 \), then the 4th column is \( 7 + 5 = 12 \), so \( Y = 2 \). We are done.

*Be sure to check the addition to ensure that everything is OK!*

The solution to this alphametic is: 9 5 6 7 SEND

1 0 8 5 MORE

1 0 6 5 2 MONEY
In “Mathematics on Vacation” Joseph Madachy presents this response to the SEND MORE MONEY problem:  

\[
\begin{array}{ccc}
A & L & S \\
L & A & S \\
N & O \\
M & O & R & E \\
C & A & S & H \\
\end{array}
\]

There is more than one solution. The question posed was: “What is the largest amount of CASH available to send?” Only one solution will answer this question. Can you find it?

Another type of logic puzzle involves lying and truth-telling. Perhaps you recall the riddle about the missionary who was about to be eaten by cannibals. They allowed him to make one final statement before cooking. He was informed that if his statement was a lie, he would be boiled. If his statement was true, he would be fried. What statement did the missionary make to save his life?

The statement had to create an impossible dilemma for the cannibals – something that made boiling or frying impossible. The missionary, after a few seconds of thought (he was a good problem solver), said “I am going to be boiled.”

- If this is a true statement, then he would be fried. But he said he’d be boiled, so frying him would make that a false statement – so frying was not an option.
- If this is a false statement, then he would be boiled. But he said he’d be boiled, so boiling him would make that a true statement – so boiling was not an option.

The cannibals, still scratching their heads, had to let the missionary go free.

The following problem also involves lies and truths. It can be solved logically in a few ways – but since the point of my article is trial and error I will take that route in my solution. Try to solve it yourself before reading my answer.

**PROBLEM #6: Pink, White and Blue**

Three different tribes lived on a remote tropical island: the Blues, the Whites and the Pinks.

- a Blue always answers a question truthfully
- a White always lies
- a Pink always alternates between lies and truths when answering a series of questions. The first response could be either T or F.

A visitor to the island approached a group of three native women whose names were Ms. White, Ms. Pink and Ms. Blue. The group was from each of the three different tribes.

Taking Ms. Pink aside, the visitor asked her three questions and received these answers:

**Visitor:** Ms. Pink, are you a Pink?  
**Ms. Pink:** Yes I am.

**Visitor:** And what tribe is Ms. White from?  
**Ms. Pink:** Ms. White is a White.

**Visitor:** So Ms. Blue is a Blue?  
**Ms. Pink:** Yes.

From this conversation, the visitor was able to determine each woman’s tribe. Can you?
LOGIC: Ms. Pink is either a Pink or she is not a Pink.

TRIAL AND ERROR: The process of trial & error involves listing all possibilities and examining each in turn to discover which is correct. Many logic puzzles involve the examination of various options, so trial & error is a legitimate way to proceed. That is the case in this solution.

<table>
<thead>
<tr>
<th>Case 1:</th>
<th>Case 2:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ms. Pink says “I am a Pink” is TRUE.</td>
<td>Ms. Pink says “I am a Pink” is FALSE.</td>
</tr>
<tr>
<td>If this is a TRUE answer (i.e. Ms. Pink is a Pink), then her 2nd answer must be FALSE and her 3rd answer must be TRUE.</td>
<td>Ms. Pink cannot be a Blue because a Blue never lies. This means that Ms. Pink must be a White.</td>
</tr>
<tr>
<td>Her 2nd answer said that Ms. White is a White. Since this must be false, Ms. White must be a Blue.</td>
<td>A White always lies, so both the 2nd and 3rd answers must be FALSE.</td>
</tr>
<tr>
<td>Her 3rd answer said that Ms. Blue is a Blue. This must be true - but this contradicts the fact that Ms. White is a Blue!</td>
<td>So Ms. White is not a White and Ms. Blue is not a Blue. Since Ms. Pink is a White, the only possibility left for Ms. Blue is Pink.</td>
</tr>
</tbody>
</table>

Because our assumption that “Ms. Pink is a Pink” led to a contradiction, our assumption that the original statement was TRUE is incorrect.

That means Ms. White is a Blue.

Solution: Ms. Pink is a White.
Ms. White is a Blue.
Ms. Blue is a Pink.

Note:

In mathematics, there is a special name given to the type of logical argument used in this solution: reductio ad absurdum. We reach a stage where there is more than one possible answer to the problem, say ‘A’ or ‘B’. We assume that ‘A’ is true – and then proceed on the basis of that assumption. If a contradiction (or false result) is achieved, then our initial assumption has been reduced to an absurd conclusion. So ‘A’ is false. Thus, ‘B’ must be true. This is an indirect method of proof – just one more perfectly valid and acceptable form of trial and error.

It's not magic; it's mathematics! We may not be Houdini, but this type of solution allows us to escape from many problems!
Here’s a similar type of puzzle for you to try on your own…

PROBLEM #7: Pink, White, Blue and Yellow  
Still on the same tropical island with the same three tribes, our intrepid visitor walked on and met three men named Mr. Blue, Mr. White and Mr. Pink. The group included a representative from each tribe. Shortly after a fourth man, Mr. Yellow, arrived on the scene. Our visitor asked the first three men the same two questions:

1. What is your tribe?    2. What is Mr. Yellow’s tribe?

These were the answers he received:

Mr. Blue:   I’m not a Blue.   Mr. Yellow is a White.
Mr. White:  I’m not a White.  Mr. Yellow is a Pink.
Mr. Pink:   I’m not a Pink.   Mr. Yellow is a Blue.

From these answers our puzzle-solving visitor was able to determine Mr. Yellow’s tribe. Can you?

**Hint:** For each man, make up a chart with the three possible tribes he could be from – and then test each possibility in turn. You should find that Mr. Yellow is a Blue.

Good luck with the puzzles.

PART 2 of this article will appear in E-Magazine Issue #34 on March 1st.

**Dave Didur** is a retired secondary school mathematics teacher with a B. Sc. degree from the University of Toronto majoring in Mathematics and Physics. He was Head of Mathematics for over twenty years, as well as the Computer Co-ordinator and consultant for the Board of Education for the City of Hamilton. He served with the Ontario Ministry of Education for three years as an Education Officer.

This article is the tenth of a series of mathematics articles published by CHASA.

Marvellous Mathematics – Introduction
Euclidian Geometry – Article #1
Non-Euclidean Geometry – Article #2
Rational Numbers – Fractions, Decimals and Calculators – Article #3
Continued Fractions – Article #4
Introduction to Fractals: The Geometry of Nature – Article #5
Solving Algebraic Equations (One Variable) – Article #6
Solving Systems of Equations in Two Variables – Article #7
Calculating the Value of Pi – Article #8
Xmas Xmath – Article #9

CHASA has received many communications from concerned parents about the difficulties their children are having with the math curriculum in their schools as well as their own frustration in trying to understand the concepts - so that they can help their children. The intent of these articles is to not only help explain specific areas of history, concepts and topics in mathematics, but to also show the beauty and majesty of the subject.
References *(these are some of my puzzle books)*


The following puzzle is from this book:

*Trial & error* is used in the solution of common ciphers, or coded messages, in which each letter of the alphabet is represented by another letter. First, the message is examined in order to find which letters occur most frequently. In the English language, the letters E T A I O S and H occur most often. It is logical to begin with these substitutions – and to use trial & error from there. It can be a long, tedious process.

Here’s a cipher to solve. It’s a famous Shakespearian quotation.

SANTA BE AS SANT SUPS MOSU TYRTOS MAE

*(but the spaces are not in the correct places between words)*

The following two-word clue is also written in the same cipher:

UPLIFTS OAIMIAYRD

*(the space is correctly located here between two words)*


5. “Mathematics on Vacation” by Joseph S. Madachy, Thomas Nelson and Sons Ltd., Don Mills ON Canada, 1968